Optimal Fault Detection for Closed-Loop Linear Uncertain Systems

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***Abstract*— Robust fault detection is crucial for ensuring the reliability and safety of complex engineering systems. However, distinguishing faults from disturbances and model uncertainty which are inherently present in any practical system remains remains a challenging task. This paper addresses the robust fault detection filter design problem for continuous-time linear time-invariant uncertain systems operating in open or closed-loop configurations. The proposed framework offers a unified approach to handle both parametric and dynamic uncertainties by solving a single Riccati equation, based on a worst-case disturbance and uncertainty scenario. The efficacy of the proposed approach is demonstrated on a numerical multivariable double mass-spring-damper system. The results illustrate that an optimal compromise is achieved between fault sensitivity and rejection of modelling uncertainties and disturbances. This capability enables the clear differentiation between faults and undesired effects in the residuals, thereby enhancing fault detection reliability, ultimately contributing to improved safety and performance.**

I. INTRODUCTION

Fault diagnosis systems are indispensable in modern engineering systems, which are becoming increasingly complex. These systems span from high-precision production equipment to aerospace applications, all of which necessitate an increased focus on improved system safety and reliability. Given the inherent variability and uncertainty in real-world operating environments, the question is not whether a machine will fail without proper monitoring and maintenance, but rather, when it will fail. Therefore, timely detection and identification of faults is crucial to mitigate the risk of performance degradation, disasters, and loss of human life or significant damages. The knowledge gained from diagnostic systems can be used to schedule maintenance optimally, thereby reducing both risk and downtime. In this context, effective fault detection methodologies play a vital role in ensuring the reliability and safety of complex engineering systems.

Driven by the growing demand for more safe and reliable systems, the field of fault diagnosis has gained much attention over the past decades. Significant progress has been made using model-based methods [1], [2], [3], [4], [5]. Particularly, observer-based methods as considered in this paper have gained a significant amount of attention and have been proven to be effective in detecting various types of faults. Building on these methods, extensive strategies have been developed for fault detection and isolation.

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However, distinguishing faults from unknown disturbances and noise, inherently present in any practical system, is not evident. It is widely acknowledged that achieving satisfactory performance in model-based fault diagnosis systems requires a delicate balance between fault sensitivity and disturbance rejection [6]. Several robust fault diagnosis methods have been developed for this purpose, including factorization-based techniques [6], often implemented through the solution of a Riccati equation [7]. Alternatively, $\mathcal{H}_-/\mathcal{H}_\infty$ techniques have been utilized, employing Linear Matrix Inequality (LMI) synthesis [8], [9]. These methods are optimal in the sense that the residual is as sensitive to faults as possible provided that the disturbance and plant model are exactly known.

Unfortunately, models of real-world systems are inherently imperfect. This modelling uncertainty stems from factors such as limited estimation accuracy, simplifications and assumptions, or system variability. Robust methods have been developed to address this uncertainty, such as μ-synthesis [10]. Alternatively, approaches like $\mathcal{H}_-/\mathcal{H}_\infty$ synthesis [11], [12] are utilized, sometimes in combination with μ_g-analysis [13], or $\mathcal{H}_\infty$ model-matching techniques incorporating Linear Matrix Inequality (LMI) synthesis [14].

The majority of these methods accounting for modelling uncertainty tend to be either overly conservative or difficult to apply. Moreover, some lack guaranteed optimality or lack simplicity. E.g., the methods are not tailored to closed-loop systems or not easily extendable for fault detection and isolation purposes.

Although several important steps have been taken towards fault diagnosis for complex systems, at present a direct method for fault detection in uncertain closed-loop systems is lacking. This paper aims to address this lack by presenting an optimal $\mathcal{H}_i/\mathcal{H}_\infty$ solution to the fault detection filter design problem for continuous-time linear time-invariant (LTI) multi-input multi-output (MIMO) uncertain closed-loop systems, subjected to additive disturbances and faults. The solution, solved using a single Riccati equation, is relatively simple, intuitive, and achieves an optimal compromise between the rejection of disturbances and modeling uncertainty with respect to fault sensitivity.

The paper is organized as follows: After preliminaries, the fault detection filter optimization problem is formulated, and an optimal solution is proposed. Subsequently, a numerical example on a MIMO double mass-spring-damper system is presented, followed by a summary of key findings.

II. PRELIMINARIES

The sets of real numbers and nonnegative real numbers are indicated by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$. The Euclidean norm is denoted by $\|\cdot\|$, while the absolute value is represented by $|\cdot|$.

The maximum and minimum singular values of the matrix A are denoted by $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively. The real rational subspace of $\mathcal{H}_\infty$ is denoted by $\mathcal{RH}_\infty$. $y \in \mathcal{L}_2$ if $\|y\|_2^2 = \int_0^\infty y^\top(t)y(t)\mathrm{d}t < \infty$. $y \in \mathcal{L}_{2e}$ if $\|y\|_{2T}^2 = \int_0^T y^\top(t)y(t)\mathrm{d}t < \infty$, $T \in \mathbb{R}_{\geq 0}$. A transfer function N is called *inner* if $N \in \mathcal{RH}_\infty$ and $N^H N = I$ and co-inner if $N \in \mathcal{RH}_\infty$ and $NN^H = I$. A transfer function M is called *outer* if $M \in \mathcal{RH}_\infty$ and has full row normal rank and has no open right half plane zeros.

The following definitions [7], [9], [15] and matrix factorization are used in this paper [4], [16]

Definition 1: (Minimum gain) The smallest gain of the continuous-time LTI system $G : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, that is the $\mathcal{H}_-$ index, is defined as

$$\|G\|_- = \inf_{\omega \in \mathbb{R}_{\geq 0}} \underline{\sigma}(G(j\omega)).$$

The minimum gain is not a norm and therefore named the $\mathcal{H}_-$ index.

Definition 2: (Maximum gain) The $\mathcal{H}_\infty$ norm of the continuous-time LTI system $G : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, denoted as $\|G\|_\infty$, is given by

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}_{\geq 0}} \bar{\sigma}(G(j\omega)).$$

Lemma 1: (Left Coprime Factorization) Let $G(s)$ be a proper real rational transfer matrix. A left coprime factorization (LCF) of system G is a factorization $G = M^{-1}N$, where M and N are left-coprime over $\mathcal{RH}_\infty$. Let $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a detectable state-space realization of G and let L be a matrix with appropriate dimensions such that $A + LC$ is stable, then a left coprime factorization of G is given by state space representation

$$\begin{bmatrix} M & N \end{bmatrix} := \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array}\right] \in \mathcal{RH}_\infty.$$

Lemma 2: (Co-inner-outer factorization) Let $G \in \mathcal{RH}_\infty$ be a transfer matrix of $p \times m$ and assume $p \leq m$. Then there exists an LCF $G = \tilde{M}^{-1}\tilde{N}$ such that $\tilde{N}$ is co-inner and $\tilde{M}$ is co-outer if and only if $GG^H > 0$ on the $j\omega$-axis, including at ∞. This factorization is unique up to a constant unitary multiple. Assume that $G = \left[\begin{array}{c|c} A - j\omega I & B \\ \hline C & D \end{array}\right]$ is detectable and that $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ has full row rank for all $\omega \in \mathbb{R}_{\geq 0}$. Then, a particular realization of the desired co-inner-outer factorization (CIOF) is

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} := \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline R^{-1/2}C & R^{-1/2} & R^{-1/2}D \end{array}\right] \in \mathcal{RH}_\infty,$$

where $R = DD^\top > 0$, and $L = -\left(BD^\top + YC^\top\right)^\top R^{-1}$ and $Y \geq 0$ be the stabilizing solution to the Riccati equation

$$\left(A - BD^\top R^{-1}C\right)Y + Y\left(A - BD^\top R^{-1}C\right)^\top -$$
$$YC^\top R^{-1}CY + B\left(I - D^\top R^{-1}D\right)B^\top = 0,$$

such that $A - BD^\top R^{-1}C - YC^\top R^{-1}C$ is stable.

Definition 3: (Linear fractional transformation) For matrices N and $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ of appropriate partitioning, the lower linear fractional transformation (LFT) is defined as $\mathcal{F}_l(M, N) = M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21}$ and the upper LFT as $\mathcal{F}_u(M, N) = M_{22} + M_{21}N(I - M_{11}N)^{-1}M_{12}$, under the assumption that the involved matrix inverses exists.

III. Problem Formulation

Consider the output $y \in \mathbb{R}^{n_y}$ of the uncertain LTI processes described by

$$y = G_u(s,\Delta)u + G_d(s,\Delta)d + G_f(s,\Delta)f, \qquad (1)$$

where $G_u(s,\Delta)$, $G_d(s,\Delta)$ and $G_f(s,\Delta)$ are uncertain transfer function matrices from the control input $u \in \mathbb{R}^{n_u}$, the disturbance $d \in \mathbb{R}^{n_d}$, and the fault $f \in \mathbb{R}^{n_f}$. The modelling uncertainty is denoted by $\Delta \in \mathbf{\Delta}$ and can be parametric or dynamic with suitable dimensions. Throughout the remainder of this work, the Laplace operator s is omitted when clear from context. The system (1) is controlled to follow a reference $r \in \mathbb{R}^{n_y}$ by means of a robustly stabilizing feedback controller $K(s)$, i.e., $u = K(r - y)$, see Figure 1. Substitution of the feedback relation into (1) results in the closed-loop input-output relation for uncertain systems

$$y = S_\Delta\left(G_u(s,\Delta)Kr + G_d(s,\Delta)d + G_f(s,\Delta)f\right), \qquad (2)$$

where $S_\Delta = (I + G_u(s,\Delta)K)^{-1}$ is the uncertain sensitivity function.

The closed-loop system is augmented with a fault detection system which is connected to the control input u and output y. The fault detection system generates residual signals $\epsilon \in \mathbb{R}^{n_y}$ that enable to detect faults f, despite the influence of the external disturbances r and d. It has been demonstrated in [17] that all residual generators can be parameterized as

$$\epsilon = R(s)\left(\tilde{M}_u(s)y - \tilde{N}_u(s)u\right), \qquad (3)$$

where $R(s) \in \mathcal{RH}_\infty^{n_y \times n_y}$ is a post-filter of the pre-residual

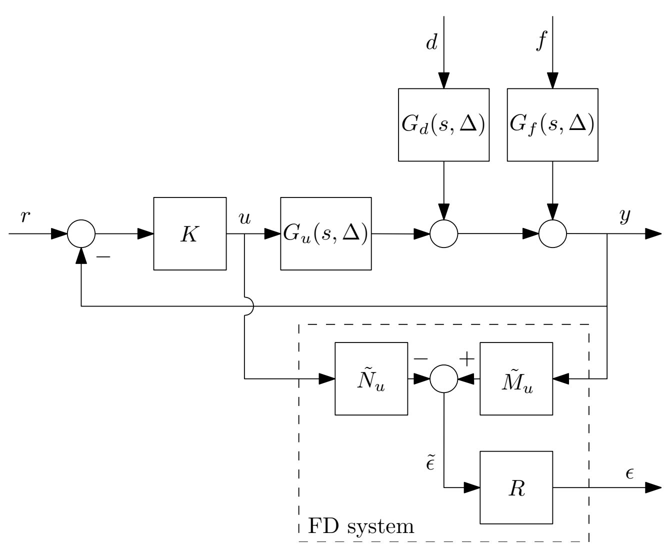

Fig. 1. Generic fault detection configuration for uncertain closed-loop controlled systems. The control input u and output y form the inputs for the fault detection (FD) system which generates the residual signal ϵ.

$\tilde{\epsilon} \in \mathbb{R}^{n_y}$, and transfer function matrices $\tilde{M}_u(s), \tilde{N}_u(s) \in \mathcal{RH}_\infty$ are a left coprime factorization of the nominal plant $G_u(s,0)$, i.e., $G_u(s,0) = \tilde{M}_u^{-1}\tilde{N}_u$. The uncertain feedback system (2) augmented with the fault detection system (3) is graphically depicted in Figure 1.

Let the uncertain part of the plant $G_u(s,\Delta)$ be defined as

$$\tilde{G}_u(s,\Delta) := G_u(s,\Delta) - G_u(s,0). \tag{4}$$

Substitution of (2), the control law $u = K(r - y)$, and (4) into (3), gives the residual dynamics for uncertain closed-loop systems as

$$\begin{aligned}\epsilon = R\tilde{M}_u\{&\underbrace{\tilde{G}_u(s,\Delta)K\left(I - S_\Delta G_u(s,\Delta)K\right)}_{T^\Delta_{\tilde{\epsilon}r}} r + \\ &(G_d(s,\Delta) - \underbrace{\tilde{G}_u(s,\Delta)KS_\Delta G_d(s,\Delta)}_{T^\Delta_{\tilde{\epsilon}d}})d + \\ &(G_f(s,\Delta) - \underbrace{\tilde{G}_u(s,\Delta)KS_\Delta G_f(s,\Delta)}_{T^\Delta_{\tilde{\epsilon}f}})f\}.\end{aligned} \tag{5}$$

To enhance readability, let $T_{\tilde{\epsilon}r}(s,\Delta) = \tilde{M}_u T^\Delta_{\tilde{\epsilon}r}$, $T_{\tilde{\epsilon}d}(s,\Delta) = \tilde{M}_u(G_d - T^\Delta_{\tilde{\epsilon}d})$ and $T_{\tilde{\epsilon}f}(s,\Delta) = \tilde{M}_u(G_f - T^\Delta_{\tilde{\epsilon}f})$ be defined as the uncertain transfers from the reference r, the disturbances d, and the faults f, to the pre-residual $\tilde{\epsilon}$. Then, the residual dynamics for uncertain closed-loop systems is written as

$$\epsilon = R\underbrace{\begin{bmatrix} T_{\tilde{\epsilon}r}(s,\Delta) & T_{\tilde{\epsilon}d}(s,\Delta)\end{bmatrix}}_{\tilde{G}_d(s,\Delta)}\underbrace{\begin{bmatrix} r \\ d\end{bmatrix}}_{\tilde{d}} + RT_{\tilde{\epsilon}f}(s,\Delta)f. \tag{6}$$

The residual dynamics clearly show the inherent trade-off between sensitivity to faults and robustness against external disturbances and the effect of modelling uncertainties. The aim is to make $R\tilde{G}_d(s,\Delta)$ as small as possible for all $\Delta \in \boldsymbol{\Delta}$ to mitigate the effects of r and d on the residual, and, on the other hand, to make $RT_{\tilde{\epsilon}f}(s,\Delta)$ as large as possible for all $\Delta \in \boldsymbol{\Delta}$ to be as sensitive to faults as possible.

A natural way to evaluate the robustness against disturbances $\tilde{d}$ is through an induced norm, that is, the worst-case impact of $\tilde{d}$ on ϵ, which is described by the $\mathcal{H}_\infty$ norm. Characterizing the sensitivity of faults necessitates a more intricate approach. The singular values of a matrix give a measure for the amplification in the direction of maximum action among all directions orthogonal to the singular vectors of any larger singular value. In essence, these give a measure for amplification in the principal directions of a system. In this context, all singular values $\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega,\Delta)\right), \omega \in [0,\infty)$, $\Delta \in \boldsymbol{\Delta}$, where $i = 1,\ldots,n_\sigma$ and $n_\sigma = \min(n_y, n_f)$, together form a measure of the fault sensitivity and cover all directions of the subspace spanned by $R(j\omega)T_{\tilde{\epsilon}f}(j\omega,\Delta)$.

With the measure for worst-case disturbance amplification and the measure for fault sensitivity, the $\mathcal{H}_i/\mathcal{H}_\infty$ performance index is defined at every frequency ω and for every $i = 1,\ldots,n_\sigma$, as

$$J_{i,\omega,\Delta}(R) = \frac{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega,\Delta)\right)}{\|R(s)\tilde{G}_d(s,\Delta)\|_\infty}, \tag{7}$$

where $\Delta \in \boldsymbol{\Delta}$. The objective is to find the fault detection filter that maximizes this ratio for all singular values i, and at every frequency ω, while accounting for the uncertainty $\Delta \in \boldsymbol{\Delta}$.

Problem 1: Consider the residual dynamics (5) of a closed-loop uncertain system described by (2) and let $\gamma > 0$ be a user-defined combined disturbance and uncertainty rejection level. Determine a stable transfer function matrix $R(s) \in \mathcal{RH}_\infty^{n_y \times n_y}$ which maximizes

$$\sup_{R(s)\,\in\,\mathcal{RH}_\infty}\left\{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega,\Delta)\right) \mid \|R(s)\tilde{G}_d(s,\Delta)\|_\infty \leq \gamma\right\}, \tag{8}$$

for $i = 1,\ldots,n_\sigma, \omega \in [0,\infty)$, $\Delta \in \boldsymbol{\Delta}$.

Remark 1: Note that the introduction of $\gamma > 0$ has no effect on the performance index $J_{i,\omega,\Delta}(R)$ in (7). Since the filter $R(j\omega)$ can be scaled arbitrarily, the bound γ merely serves as a scaling parameter that has no influence on the optimal ratio, but ensures that the optimal solution for optimization problem (8) is unique.

Remark 2: Note that the optimization problem (8) is multiobjective since the solution solves for all $i = 1,\ldots,n_\sigma$. Hence, the optimization problem includes the special $\mathcal{H}_\infty/\mathcal{H}_\infty$ and the $\mathcal{H}_-/\mathcal{H}_\infty$ objectives

$$\sup_{R(s)\,\in\,\mathcal{RH}_\infty}\left\{\|R(s)T_{\tilde{\epsilon}f}(s,\Delta)\|_\infty \mid \|R(s)\tilde{G}_d(s,\Delta)\|_\infty \leq \gamma\right\}, \tag{9}$$

and

$$\sup_{R(s)\,\in\,\mathcal{RH}_\infty}\left\{\|R(j\omega)T_{\tilde{\epsilon}f}(j\omega,\Delta)\|_- \mid \|R(s)\tilde{G}_d(s,\Delta)\|_\infty \leq \gamma\right\}, \tag{10}$$

respectively.

Remark 3: Although the framework presented in this study focuses on uncertain closed-loop systems, the same methodology and solution are equally applicable to open-loop uncertain systems. This merely necessitates deriving the residual dynamics in (6) for uncertain open-loop systems, yielding a residual ϵ that depends on the input u rather than the reference r.

Remark 4: Note that when there is no uncertainty, i.e., $\Delta = 0$ and thus $T^\Delta_{\tilde{\epsilon}r} = 0, T^\Delta_{\tilde{\epsilon}d} = 0$, and $T^\Delta_{\tilde{\epsilon}f} = 0$, the residual dynamics in (6) simplify to the same expression for closed (and open)-loop nominal systems, that is

$$\epsilon = R\tilde{M}_u(G_d d + G_f f).$$

Hence, the residual only depends on disturbances d and possible faults f, and is independent of the reference r.

IV. Robust Fault Detection Filter Design

In this section, the solution to the robust fault detection filter optimization problem (8) is presented. First, an upper bound is established to encapsulate the worst-case scenario of uncertain exogenous disturbances. Subsequently, this upper bound is used to solve the optimal robust fault detection filter design problem, followed by several remarks.

Definition 4: Let $\tilde{d}_1$ and $\tilde{d}_2$ be unitary input vectors, i.e., $\|\tilde{d}_1\|_2 = \|\tilde{d}_2\|_2 = 1$, then $G_d(s) \in \mathcal{RH}_\infty^{n_y \times (n_y+n_d)}$ is an upper-bound envelope of $\tilde{G}_d(s,\Delta)$ if for all vectors with $\|\tilde{d}_2\|_2 = 1$ the following condition is satisfied

$$\|\tilde{G}_d(j\omega,\Delta)\tilde{d}_1\|_2 \leq \|G_d(j\omega)\tilde{d}_2\|_2 \quad \forall\omega, \forall\Delta \in \boldsymbol{\Delta}, \tag{11}$$

where $\tilde{d}_1$ a unitary vector such that the output of the particular realization of $\tilde{G}_d(j\omega, \Delta)\tilde{d}_1$ matches the direction of $G_d(j\omega)\tilde{d}_2$.

Definition 5: Let $\tilde{d}_1$ and $\tilde{d}_2$ be unitary input vectors, then $G_d(s) \in \mathcal{RH}_\infty^{n_y \times (n_y+n_d)}$ is a tight upper-bound envelope of $\tilde{G}_d(s, \Delta)$ if for all $\|\tilde{d}_2\|_2 = 1$,

$$\max_{\Delta \in \boldsymbol{\Delta}} \|\tilde{G}_d(j\omega, \Delta)\tilde{d}_1\|_2 = \|G_d(j\omega)\tilde{d}_2\|_2 \quad \forall \omega, \tag{12}$$

where $\tilde{d}_1$ a unitary vector such that the output of the particular realization of $\tilde{G}_d(j\omega, \Delta)\tilde{d}_1$ matches the direction of $G_d(j\omega)\tilde{d}_2$.

Hence, a tight upper bound implies that the worst-case amplification of $\tilde{G}_d(j\omega, \Delta)$ is equal to the amplification of $G_d(j\omega)$ at every ω. Next, the complete solution to the robust fault detection filter design problem is presented.

Theorem 1: Suppose the following assumptions are satisfied

- (A_1) Let $G_d = (A, B_d, C, D_d)$ be written as a state space with A Hurwitz and (A, C) detectable.
- (A_2) D_d has full row rank.
- (A_3) $G_d(s)$ has no transmission zeros on the imaginary axis.
- (A_4) Let $G_d(s)$ be a tight upper bound realization of $\tilde{G}_d(s, \Delta)$.

Then there exists a co-inner-outer factorization of $G_d(s) = G_{do}(s)G_{di}(s)$ and let the optimal fault detection filter which maximizes (8) be parameterized as

$$\epsilon = R_{\text{opt}} \begin{bmatrix} \tilde{M}_u & -\tilde{N}_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{13}$$

with the optimal post-filter $R_{\text{opt}}(s) = \gamma G_{do}^{-1}$ and $\tilde{M}_u, \tilde{N}_u \in \mathcal{RH}_\infty$ any LCF of the nominal system $G_u(s, 0)$. This filter achieves for all ω, $\Delta \in \boldsymbol{\Delta}$, $i = 1, \ldots, n_\sigma$

$$\sup_{R(s) \in \mathcal{RH}_\infty} J_{i,\omega,\Delta}(R_{\text{opt}}) = \sigma_i \left(G_{do}^{-1}(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta) \right),$$

where the corresponding state-space representation of R_{opt} is given by

$$R_{\text{opt}}(s) = \gamma \left[\begin{array}{c|c} A + L_0 C & L_0 \\ \hline R_d^{-1/2} C & R_d^{-1/2} \end{array} \right] \in \mathcal{RH}_\infty, \tag{14}$$

in which $R_d := D_d D_d^\top > 0$ and $Y \geq 0$ is the stabilizing solution to the Riccati equation

$$\left(A - B_d D_d^\top R_d^{-1} C\right) Y + Y \left(A - B_d D_d^\top R_d^{-1} C\right)^\top - YC^\top R_d^{-1} CY + B_d \left(I - D_d^\top R_d^{-1} D_d\right) B_d^\top = 0,$$

such that $A - B_d D_d^\top R_d^{-1} C - YC^\top R_d^{-1} C$ is stable and

$$L_0 := -\left(B_d D_d^\top + YC^\top\right)^\top. \tag{15}$$

Proof: From (11) follows that

$$\|\tilde{G}_d(s, \Delta)\|_\infty \leq \|G_d(s)\|_\infty \quad \forall \Delta \in \boldsymbol{\Delta}.$$

Pre-multiplication by the post-filter $R(s) \in \mathcal{RH}_\infty$, shows that

$$\|R(s)\tilde{G}_d(s, \Delta)\|_\infty \leq \|R(s)G_d(s)\|_\infty \quad \forall \Delta \in \boldsymbol{\Delta}.$$

Substitution into (7) gives

$$J_{i,\omega,\Delta}(R) = \frac{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|R(s)\tilde{G}_d(s, \Delta)\|_\infty} \geq \frac{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|R(s)G_d(s)\|_\infty}. \tag{16}$$

Assuming a tight upper bound as (A_4) gives that

$$\|R(s)\tilde{G}_d(s, \Delta)\|_\infty = \|R(s)G_d(s)\|_\infty,$$

which results in equality in (16) as

$$J_{i,\omega,\Delta}(R) = \frac{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|R(s)\tilde{G}_d(s, \Delta)\|_\infty} = \frac{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|R(s)G_d(s)\|_\infty}. \tag{17}$$

From (A_1) to (A_3) follows that a CIOF of $G_d(s)$ exists. I.e., $G_d(s) = G_{do}(s)G_{di}(s)$, where $G_{di}(s)$ is the co-inner matrix satisfying $G_{di}(j\omega)G_{di}^\top(-j\omega) = I$ and having $\sigma(G_{di}(j\omega)) = I$ for all ω, and $G_{do}(s)$ is co-outer satisfying $G_{do}^{-1} \in \mathcal{RH}_\infty$ and thus $G_{do}(s)^{-1}G_{do}(s) = I$. Now parameterize $R(s)$ as

$$R(s) = \gamma Q(s) G_{do}^{-1}(s),$$

where $Q(s) \in \mathcal{RH}_\infty$ an arbitrary stable TFM. Substitution into (17) yields

$$\begin{aligned} &J_{i,\omega,\Delta}(R) = \\ &\frac{\sigma_i\left(R(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|R(s)G_d(s)\|_\infty} = \frac{\sigma_i\left(\gamma Q(j\omega)G_{do}^{-1}(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|\gamma Q(s)G_{di}(s)\|_\infty} \\ &= \frac{\sigma_i\left(Q(j\omega)G_{do}^{-1}(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right)}{\|Q(s)\|_\infty} \\ &\leq \sigma_i\left(G_{do}^{-1}(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right), \end{aligned} \tag{18}$$

where it is used that $G_{di}(j\omega)G_{di}^\top(-j\omega) = I$ so that $\|Q(s)G_{di}(s)\|_\infty = \|Q(s)\|_\infty$.

From the inequality in (18) follows that setting $R_{\text{opt}}(s) = \gamma G_{do}^{-1}(s)$, i.e., $Q(s) = I$ gives the optimal performance. Hence, it holds that for all ω, $\Delta \in \boldsymbol{\Delta}$, $i = 1, \ldots, n_\sigma$

$$J_{i,\omega,\Delta}(R_{\text{opt}}) = \sigma_i\left(G_{do}^{-1}(j\omega)T_{\tilde{\epsilon}f}(j\omega, \Delta)\right). \tag{19}$$

resulting in the optimal solution of Problem 1 for uncertain closed-loop systems. The state-space realization of the outer term $G_{do}^{-1}(s)$ follows directly from Lemma 2. ■

Next, a series of observations will be presented concerning the assumptions (A_1) to (A_3). Following this, an interpretable explanation of the optimal filter is given and an analysis is provided regarding the implications of the derived outcome with respect to assumption (A_4).

Remark 5: While it might appear that requiring A to be Hurwitz in Assumption (A_1) is restrictive, this is not the case. In fact, the system described by the transfer matrix $G_d(s)$ is always stable, as it characterizes the closed-loop transfer $\tilde{d} \to \tilde{\epsilon}$.

Remark 6: Assumption (A_2) implies that $n_y \leq n_d$ and every output is subject to some form of disturbance or measurement noise. It can be argued that this assumption can be made without loss of generality, as it is practically impossible to obtain perfect measurements in any system. Additionally, it is reasonable to assume that the measurement noise is independent of each other. Hence, it is reasonable to assume that the disturbance matrix D_d has full row rank. Readers are referred to [7] for an adaptation of the state-space matrices in case D_d is not full row rank.

Remark 7: In numerous applications, Assumption (A_3) is not considered restrictive, as many systems inherently include a certain level of damping, resulting in zeros positioned away from the imaginary axis. However, in cases where Assumption (A_3) is not satisfied, alternative approaches are available if $G_d(s)$ features zeros on the imaginary axis. Readers are directed to [18] for further details on these non-standard approaches.

The following two remarks relate to the solution itself.

Remark 8: Note that the achieved performance $J_{i,\omega,\Delta}(R)$ is independent on the choice of observer gain matrices L in the LCF of the nominal plant $G(s, 0)$ used in the filter design, see Lemma 1.

Remark 9: Note that the solution is completely determined by the uncertainty and disturbance models encaptured in $\tilde{G}_d(s, \quad)$ and is therefore independent of the fault model $\tilde{G}_f(s, \quad)$.

In order to interpret the optimal post-filter, $R(s) = \gamma G_{do}^{-1}(s)$, recall that the co-outer of a TFM can be interpreted as a frequency domain magnitude profile. Hence, the worst-case influence of the exogenous disturbance $\tilde{d}$ on the residuals ϵ is homogenized across the entire subspace by the inverse of this profile. As a consequence, the optimal solution affects the transfer from faults f to residuals ϵ by an inverse weighting of this magnitude profile.

To attain the supremum as defined in (8) and thereby achieve optimal fault detection performance, it is imperative that the upper bound realization satisfies (A_4). This condition is inherently met if $G_d(s) \in \tilde{G}_d(s, \quad)$ and (11) holds for all frequencies ω and uncertainties $\quad \in \quad$. In such instances, the worst-case realization consistently bounds the other realizations from above across all frequencies. However, if the upper bound complies with (11) but is not tight, conservatism is introduced resulting in a suboptimal filter in view of (8). In this case, the result is merely optimal given the conservative upper bound. To mitigate this conservatism, which is beyond the scope of this paper, an iterative process of tightening the upper bound $G_d(s)$ can be employed. To this end, the structured singular value test is used which quantifies the discrepancy between $T_{\epsilon\tilde{d}}$ and γ, serving as a measure of the residual conservatism.

Deriving time-domain bounds for residual evaluation is beyond the scope of this paper, however, applying the optimal solution $R_{\text{opt}}(s)$ yields the residual signals

$$\epsilon = \gamma G_{do}^{-1}\tilde{G}_d(s, \quad)\tilde{d} + \gamma G_{do}^{-1}T_{\tilde{\epsilon}f}(s, \quad)f. \tag{20}$$

Note that $\|G_{do}^{-1}\tilde{G}_d(s, \quad)\|_\infty \leq \|G_{di}\|_\infty$. Then, suppose the exogenous disturbances $\tilde{d}$ are energy bounded signals, i.e., signals with bounded $\mathcal{L}_2$ norms, it follows from the expression in (20) in the fault-free case, i.e., $f = 0$, that $\|\epsilon\|_2 \leq \gamma\|\tilde{d}\|_2$ due to the norm preserving properties of a co-inner [16].

Next, the optimal filter is computed for an illustrative numerical example and the time-domain responses for fault detection are shown.

V. NUMERICAL EXAMPLE

Consider the MIMO double mass-spring-damper system in Figure 2. The system is equipped with two force inputs and provides output signals corresponding to the positions of both masses. The output measurements are perturbed by disturbances d_1 and d_2 respectively and are both susceptible to a single fault f. The spring that connects the first mass to the environment and the damper in between both masses both contain 5% uncertainty. The exact values that characterize the system are $m_1 = 0.155$ [kg], $m_2 = 0.095$ [kg], $k_1 = 7.94 \cdot 10^{-2} \pm 5\%$ [N/m], $k_2 = 2.5$ [N/m], $c_1 = 1.45 \cdot 10^{-3}$ [Ns/m], and $c_2 = 1.5 \cdot 10^{-4} \pm 5\%$ [Ns/m]. The system is robustly stabilized by dynamic feedback controller $K(s)$ with a bandwidth of 26 Hz, which is obtained through $\mathcal{H}_\infty$-synthesis.

Consider Theorem 1 with the implementable parameterization (13). Since system G_u is open-loop stable, the coprime factors of $G_u(s, 0)$ are chosen to be $\tilde{M}_u = I$ and $\tilde{N}_u = G_u(s, 0)$. The considered disturbance model has two inputs d_1 and d_2 and has a flat spectrum with a gain of -40 dB, i.e., $G_d = 0.01I$. The considered fault model G_f has a single input f acting on both outputs with a flat spectrum of 0 dB, i.e., $G_f = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$. These basic disturbance and fault models are selected for illustrative purposes, aimed at emphasizing the proposed method's effectiveness in addressing modeling uncertainty.

Following the procedure proposed in Theorem 1 with $\gamma = 1$ results in the optimal post-filter $R_{\text{opt}}(s)$ shown in Figure 3. This result is optimal since a tight upper bound exists, where $G_d(s) \in \tilde{G}_d(s, \quad)$ while (11) is satisfied. Furthermore, the singular values of transfer $T_{\epsilon\tilde{d}}$ are shown in Figure 4. Clearly, the design constraint $\|R(s)\tilde{G}_d(s, \quad)\|_\infty \leq \gamma$ is satisfied for all $\quad \in \quad$ and for all ω. Note that at each frequency ω, one of the realizations approaches the constraint $\gamma = 1$, a direct outcome of the precise upper bound $G_d(s)$. Additionally, the shape of the uncertainty envelope indicates areas of modeling uncertainty, such as the low-frequency spring characteristic and the uncertain damping of the second mode.

A time-domain simulation is performed to demonstrate the fault detection process. For this purpose, a single disturbance d_1 is applied to the output of the first mass which is modeled as $d_1 = 0.25\sin(20\pi t)$, and $d_2 = 0$. The setpoints $r_1 = r_2$ are block-signals with an amplitude equal to 0.68 and frequency $f_r = 2$ Hz starting at $T = 1$ s. A fault is injected at the systems outputs at $T = 2$ s which is a block signal with an amplitude of 0.2 and frequency $\frac{f_r}{3.5}$ Hz. The resulting

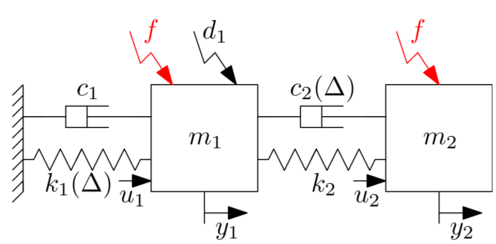

Fig. 2. Uncertain two mass-spring-damper system G_u with two inputs and two outputs. The position measurements are subjected to disturbances and a fault.